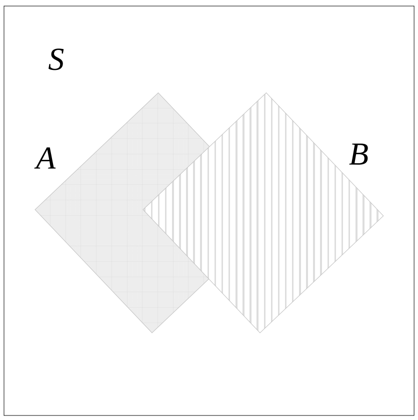
Let *S* be the set equal to the sample space. The set, *S*, contains all possible outcomes in the sample space.

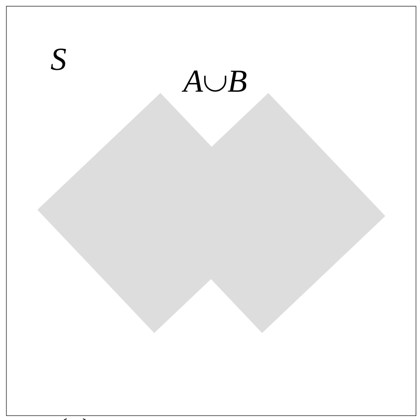
$S = \{all \quad possible \quad outcomes\}$. For example, if we flip two coins: $S = \{HH, HT, TH, TT\}$.

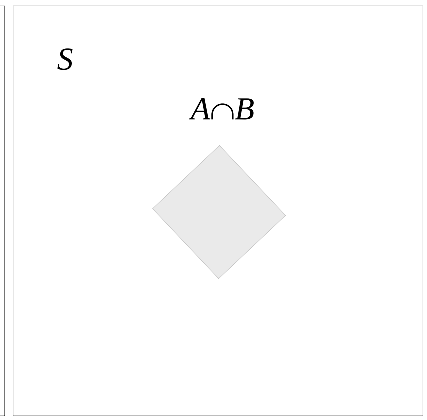
If *S* has a finite number of equally likely outcomes and we want to find the probability of an event, *E*, which is a subset of *S*, then $P(E) = \dfrac{\# \quad of \quad outcomes \quad in \quad E}{\# \quad of \quad outcomes \quad in \quad S}$.

For any sample space, *S*, (finite or infinite), the following rules apply.

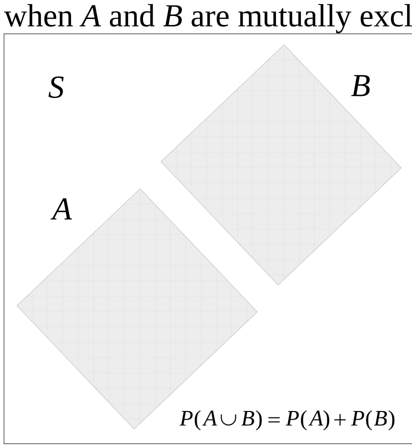
1. "And" is the same as set ***intersection*** and "or" is the same as set ***union***. So if *A* and *B* are two events (both are subset of *S*), then $P(A \quad and \quad B) = P(A \cap B)$ and $P(A \quad or \quad B) = P(A \cup B)$.

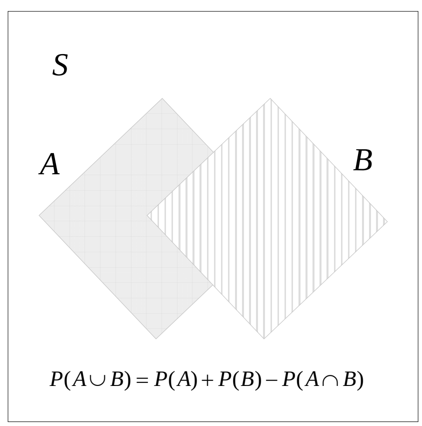




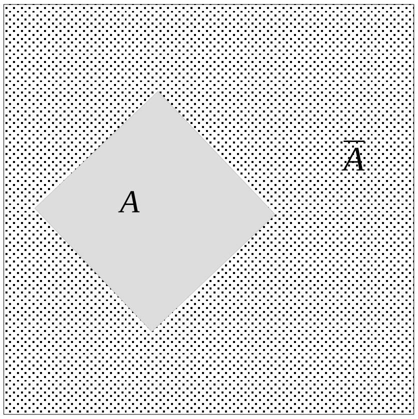


2. The empty set is written as $\varnothing$ or as $\{ \}$. $P(\varnothing) = 0$ and $P(S) = 1$.
3. Two events, *A* and *B* (both are subset of *S*), are mutually exclusive if and only if $A \cap B = \varnothing$. If *A* and *B* are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$. If you don't know that *A* and *B* are mutually exclusive, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$; (notice that the last term, $P(A \cap B)$, is zero when *A* and *B* are mutually exclusive).





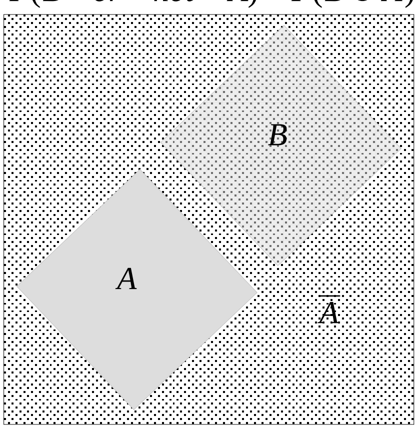
4. If A is an event (a subset of S), the compliment of A is written $\overline{A}$ (or A' or $\tilde{A}$) and $A \cup \overline{A} = S$ and $A \cap \overline{A} = \varnothing$. So, $1 = P(S) = P(A \cup \overline{A}) = P(A) + P(\overline{A})$ or $P(\overline{A}) = 1 - P(A)$.



5. If A and B are independent events then $P(A \cap B) = P(A) \cdot P(B)$. If you don't know that A and B are independent, then $P(A \cap B) = P(A) \cdot P(B \mid A)$, where $P(B \mid A)$ is the conditional probability of B given that A occurs. (Notice that when A and B are independent, $P(B \mid A) = P(B)$ and $P(A \mid B) = P(A)$. Also notice that (nonempty) independent events can NEVER be mutually exclusive since they must have a nonempty intersection. Note: there is no simple picture which adequately shows the difference between independent and non-independent sets; it always looks like the first diagram in part 1, above.)

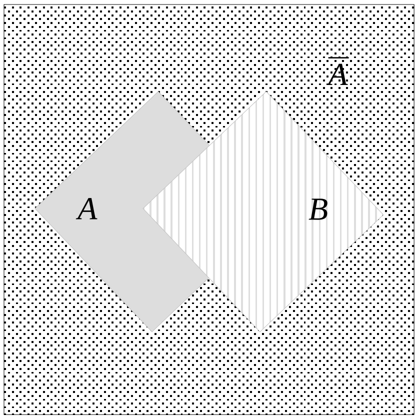
<u>Example 1</u>: If A and B are mutually exclusive events, find the probability of B or not A.

First, $A \cap B = \varnothing$, so $\overline{A} \cap B = B$ implies B and not A are NOT mutually exclusive. Then,
$P(B \quad or \quad not \quad A) = P(B \cup \overline{A}) = P(B) + P(\overline{A}) - P(B \cap \overline{A}) = P(B) + P(\overline{A}) - P(B) = P(\overline{A}) = 1 - P(A)$.



<u>Example 2</u>: Find the probability of B and not A, or B and A; (we can't assume independence or mutually exclusive).

We need to find $P((B \cap \overline{A}) \cup (B \cap A))$. But notice that *A* and $\overline{A}$ "partition" *B* into two parts since *A* and $\overline{A}$ split *S* into two parts: the part lying in *A* and the part lying in $\overline{A}$. Thus, $(B \cap \overline{A}) \cup (B \cap A) = B$. So, $P((B \cap \overline{A}) \cup (B \cap A)) = P(B)$.



6. The problem, above, can be generalized into a very useful formula:
   If $A_1, A_2, \ldots, A_n$ are *n* mutually exclusive events whose union is the whole sample space, *S*, then for any other event, *B*, $P(B) = \sum_{i=1}^{n} P(B \mid A_i) \cdot P(A_i)$.
   The sets, $A_1, A_2, \ldots, A_n$, are said to partition the sample space, and thus *B*, also, into *n* parts. They should be chosen so that finding $P(B \mid A_i)$ is easier than finding *P*(*B*). Note: the conditions that should be checked for $A_1, A_2, \ldots, A_n$ are:
   a. For each $i \neq j$, $A_i \cap A_j = \varnothing$; and
   b. $\sum_{i=1}^{n} P(A_i) = 1$.

<u>Example 3</u>: Find the probability of rolling three 6-sided dice and having them sum up to 11.
The difficulty with this problem is having three dice instead of just two. With two dice we can construct a table to calculate the probabilities of their sum:

Die 2 \ Die 1:	1	2	3	4	5	6
SUM						
1	**2**	**3**	**4**	**5**	**6**	**7**
2	**3**	**4**	**5**	**6**	**7**	**8**
3	**4**	**5**	**6**	**7**	**8**	**9**
4	**5**	**6**	**7**	**8**	**9**	**10**
5	**6**	**7**	**8**	**9**	**10**	**11**
6	**7**	**8**	**9**	**10**	**11**	**12**

P (2 dice SUM = 11) = 2/36 = 1/18

So, the solution, (using the formula, above), involves coming up with a way to reduce the problem of three dice to a problem of two dice: choose A_1, A_2, A_3, A_4, A_5, and A_6, so that A_i is the event where the first (of the three dice) is an *i*. Then, $P(A_i) = \frac{1}{6} \Rightarrow \sum_{i=1}^{6} P(A_i) = 1$ and both conditions of the formula, above, are satisfied. The reason to choose "the value" of one die is that allows us to calculate the probability of the sum of the two remaining dice using the table. For example, if we know the first die is a 1, then in order for the SUM of all three dice to be 11, the next two dice must SUM to 10. And this probability is $\frac{3}{36}$, from the table. The probability we just calculated is the conditional probability: P(3 dice SUM = 11 | 1st die is 1); and we did it by noting that P(3 dice SUM = 11 | 1st die is 1) = P(2 dice SUM = 10) = $\frac{3}{36}$.

So, $P(B) = \sum_{i=1}^{6} P(B \mid A_i) \cdot P(A_i) \Rightarrow$ P(3 dice SUM = 11) = $\sum_{i=1}^{6} P$ (3 dice SUM = 11 | 1st die is i)·$P(A_i)$

$= \sum_{i=1}^{6} P$ (3 dice SUM = 11 | 1st die is i)$\cdot \frac{1}{6}$ =

= [P(3 dice SUM = 11 | 1st die is 1) +

+ P(3 dice SUM = 11 | 1st die is 2) + P(3 dice SUM = 11 | 1st die is 3) + P(3 dice SUM = 11 | 1st die is 4)

+ P(3 dice SUM = 11 | 1st die is 5) + P(3 dice SUM = 11 | 1st die is 6)]$\cdot \frac{1}{6}$

= [P(2 dice SUM = 10) + P(2 dice SUM = 9) + P(2 dice SUM = 8) + P(2 dice SUM = 7) +

+ P(2 dice SUM = 6) + P(2 dice SUM = 5)] $\cdot \frac{1}{6}$ =

$= \left[\frac{3}{36} + \frac{4}{36} + \frac{5}{36} + \frac{6}{36} + \frac{5}{36} + \frac{4}{36}\right] \cdot \frac{1}{6} = \frac{27}{36} \cdot \frac{1}{6} = \frac{1}{8}$. (It's more likely to get a sum of 11 with 3 dice, than 2.)